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SMALL TOPOLOGICAL COMPLETE SUBGRAPHS OF "DENSE" GRAPHS

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A graph of n vertices and $4^{t^2}n^{1+\varepsilon}$ edges contains a TK_t on at most $7t^2 \log t/\varepsilon$ vertices. This answers a question of P. Erdős.

0. Introduction

P. Erdős asked the following [2]: Is it true that $G[n, n^{1+\varepsilon}]$ contains a subgraph which is nonplanar and has at most $c(\varepsilon)$ vertices? Clearly, this problem is equivalent to finding a "small" TK_5 or $TK_{3,3}$ in dense graphs.

W. Mader proved the following basic theorem [4]: *There exists a constant* $p(t)$ *such that* $G[n, p(t)n]$ *contains a* TK_t.

He also showed $p(t) \leqq O(2^t)$. On the other hand there are many results on the girth of graphs with large minimum degree [5], [6].

As a consequence of these results we know that the minimum size of a TK_3 necessarily contained by every $G[n, n^{1+\varepsilon}]$ is roughly between $1/\varepsilon$ and $2/\varepsilon$.

What we prove is the following:

Theorem. *Every* $G[n, 4^{t^2}n^{1+\varepsilon}]$ *contains a* TK_t *of size at most* $c(\varepsilon, t) \leqq 7t^2 \log t/\varepsilon$ *for all* $t \in \mathrm{N}$ *and* $\varepsilon > 0$.

This result answers the question of Erdős and brings together the above investigations on topological subgraphs and the girth of dense graphs.

Remark. From the results on the girth of graphs it follows that the best possible bound is not smaller then $O(t^2/\varepsilon)$.

Notation. A graph G of n vertices and m edges is denoted by $G[n, m]$. TK_t denotes a topological complete graph of t vertices. $\langle D \rangle_G$ denotes the subgraph of a graph G induced by $D \subset V(G)$. $d_G(x, y)$ is the length of the shortest path between $x, y \in V(G)$.

For $a \in V(G)$ we define the *distance classes* D_i^a by $D_i^a = \{x | x \in V(G), d_G(x, a) = i\}$ for $i = 0, ..., r$. The *radius* of G is defined by

$$\mathrm{rad}\,(G) = \min_x \max_y d(x, y)$$

and an $x \in V(G)$ for which the maximum is attained is called a *centre* of G.

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1. Preliminary lemmas

Lemma 1.1. *Let* $\varepsilon>\alpha>0$ *then* $G[n, cn^{1-\varepsilon}]$ *contains a subgraph* $H[m, (1/2)cm^{1-\alpha}]$ *such that* $\operatorname{rad}(H)\leqq 1+(1/\alpha)\log(\varepsilon/(\varepsilon-\alpha))$.

Proof. We might suppose that $d_G(x)\geqq cn^{\varepsilon}$ for all $x\in V(G)$ (otherwise the deletion of x results in a graph with higher average degree). For an arbitrary $a\in V(G)$ we denote the induced subgraphs $\langle D_0^a\cup\ldots\cup D_i^a\rangle$ by H_i for $i=1, \ldots, r$. We have $|E(H_i)|\geqq(1/2)cn^{\varepsilon}|H_{i-1}|$, for all the G-neighbours of vertices in H_{i-1} are contained by H_i.

Let $l=\min\{i|(c/2)|H_i|^{1+\alpha}\leqq|E(H_i)|\}$. Setting $H=H_l$ we have to estimate $\operatorname{rad}(H)\leqq l$.

We might suppose $l\geqq 2$. For $i<l$ we have $(1/2)cn^{\varepsilon}|H_{i-1}|\leqq(c/2)|H_i|^{1+\alpha}$ therefore $|H_i|\geqq(n^{\varepsilon}|H_{i-1}|)^{1/(1+\alpha)}$. By induction we obtain

$$|H_i|\geqq n^{\varepsilon\left(\frac{1}{1+\alpha}+\ldots+\frac{1}{(1+\alpha)^i}\right)}.$$

But indeed $n\geqq|H_{l-1}|$ and therefore

$$1\geqq\varepsilon\left(\frac{1}{1+\alpha}+\ldots+\frac{1}{(1+\alpha)^{l-1}}\right)=\frac{\varepsilon}{\alpha}\left(1-\frac{1}{(1+\alpha)^{l-1}}\right),$$

which leads to $\frac{\varepsilon}{\varepsilon-\alpha}\geqq(1+\alpha)^{l-1}\geqq 2^{\alpha(l-1)}$. ∎

Lemma 1.2. *For* $a\in V(G[n, cn^{1+\varepsilon}])$ *there exists an* i *such that for* $d_i=|\langle D_i^a\cup D_{i+1}^a\rangle|$ *we have* $|E\langle D_i^a\cup D_{i+1}^a\rangle|\geqq(c/2)(d_i)^{1+\varepsilon}$.

Proof. Suppose there is no such i. Then $|E\langle D_i^a\cup D_{i+1}^a\rangle|<(c/2)(d_i)^{1+\varepsilon}<(c/2)d_in^{\varepsilon}$ for all i. Therefore

$$|E(G)|\leqq\sum_{i=0}^{r-1}|E\langle D_i^a\cup D_{i+1}^a\rangle|<cn^{\varepsilon}\left((1/2)\sum_{i=1}^{r-1}d_i\right)\leqq cn^{1+\varepsilon},$$

a contradiction. ∎

Lemma 1.3. (Erdős, Gallai [3]). $G[n, t\cdot n]$ *contains a path of at least* $2t$ *vertices.* ∎

Lemma 1.4 (Erdős [1]). *A graph* G *contains a bipartite subgraph* B *with* $|E(B)|\geqq\geqq(1/2)|E(G)|$. ∎

We make one more trivial observation which however is the key of our proof.

Observation 1.5. Let G be a bipartite graph, $a\in V(G)$ and suppose there is a path P of $2t$ vertices in $\langle D_i^a, D_{i+1}^a\rangle$. Let us denote the vertices of P_{2t} by $a_1, b_1, a_2, b_2, \ldots \ldots, a_t, b_t$. Then either all the vertices a_j or all the vertices b_j are contained by D_i^a. ∎

2. The proof of the Theorem

Let G be a graph with $|E(G)|\geqq 2^{2t(t-1)}t\cdot|G|^{1+\varepsilon}$. By Lemma 1.4 it has a bipartite subgraph H_0 with $|E(H_0)|\geqq 2^{t(t-1)-1}t\cdot|H_0|^{1+\varepsilon}$.

Let us define the constants ε_i by $\varepsilon_0=\varepsilon$ and $\varepsilon_{i+1}=\varepsilon_i-\varepsilon/2t^2$ for $i=0, ..., t(t-1)$. Indeed we have $\varepsilon_i\geqq(1/2)\varepsilon$ for all i. We define a descending series of graphs $G=G_0\geqq H_0\geqq G_1\geqq ...\geqq H_{t(t-1)-1}\geqq G_{t(t-1)}$ with $|E(H_i)|\geqq 2^{2t(t-1)-2i-1}\times$ $\times t\cdot|H_i|^{1+\varepsilon_i}$ and $|E(G_i)|\geqq 2^{2t(t-1)-2i}\cdot t\cdot .G_i.^{1+\varepsilon_i}$ as follows: For $i\geqq 1$ G_i is a subgraph of H_{i-1} with

$$\text{rad }(G_i) \leqq 1+\frac{1}{\varepsilon_i}\log\left(\frac{\varepsilon_{i-1}}{\varepsilon_{i-1}-\varepsilon_i}\right).$$

It can be chosen to have at least $2^{2t(t-1)-2i}\cdot t\cdot|G_i|^{1+\varepsilon_i}$ edges by Lemma 1.1. Let us note that $\text{rad }(G_i)\leqq 1+(2/\varepsilon)(1+2\log t)$ follows from $\varepsilon_i\geqq(1/2)\varepsilon$.

For $i\geqq 1$ H_i is a subgraph of G_i, induced by two distance classes corresponding to a centre a_i of G_i. It can be chosen to have at least $2^{2t(t-1)-2i-1}\cdot t\cdot|H_i|^{1+\varepsilon_i}$ edges by Lemma 1.2.

For the last graph of the series we have $|E(G_{t(t-1)})|\geqq t\cdot|G_{t(t-1)}|$ therefore by Lemma 1.3 this graph contains a path P of $2t$ vertices. We denote the vertices of P by $x_1, y_1, ..., x_t, y_t\cdot P$ is a subgraph of G_i moreover $P\subset H_i$ for all i. Applying 1.5 we find that either all the x_j or all the y_j are in the distance class of G_i which is nearer to the centre a_i (from the two distance classes which induce H_i).

Without loss of generality we might suppose that in at least $\binom{t}{2}$ of the $G_i\big(1\leqq i\leqq t(t-1)-1\big)$, the vertices x_j are in the distance class nearer to a_i.

In each of such G_i-s we might connect two arbitrary vertices x_u and x_v by a path $P_{u,v}$ of length at most $2\,\text{rad }(G_i)$ such that $V(P_{u,v})\cap V(G_{i+1})=\{x_u, x_v\}$. Let us choose such a path for all pairs u, v such that $1\leqq u<v\leqq t$. These paths have no common inner vertices therefore the union C of them is a TK_t subgraph of G.

The number of vertices in C is at most

$$t+\binom{t}{2}\left(2\left(1+\frac{2}{\varepsilon}(1+2\log t)\right)-1\right) \leqq 7t^2\log t/\varepsilon$$

as required. ∎

Note added in proof. As E. Szemerédi informed us $c(\varepsilon, t)$ can probably be improved to the optimal $O(t^2/\varepsilon)$ using his regularity lemma.

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